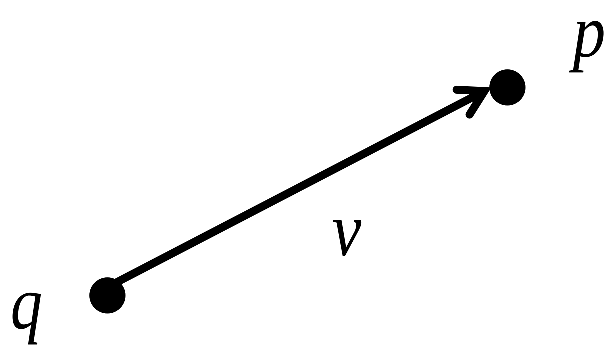
Day 02

Spatial Descriptions

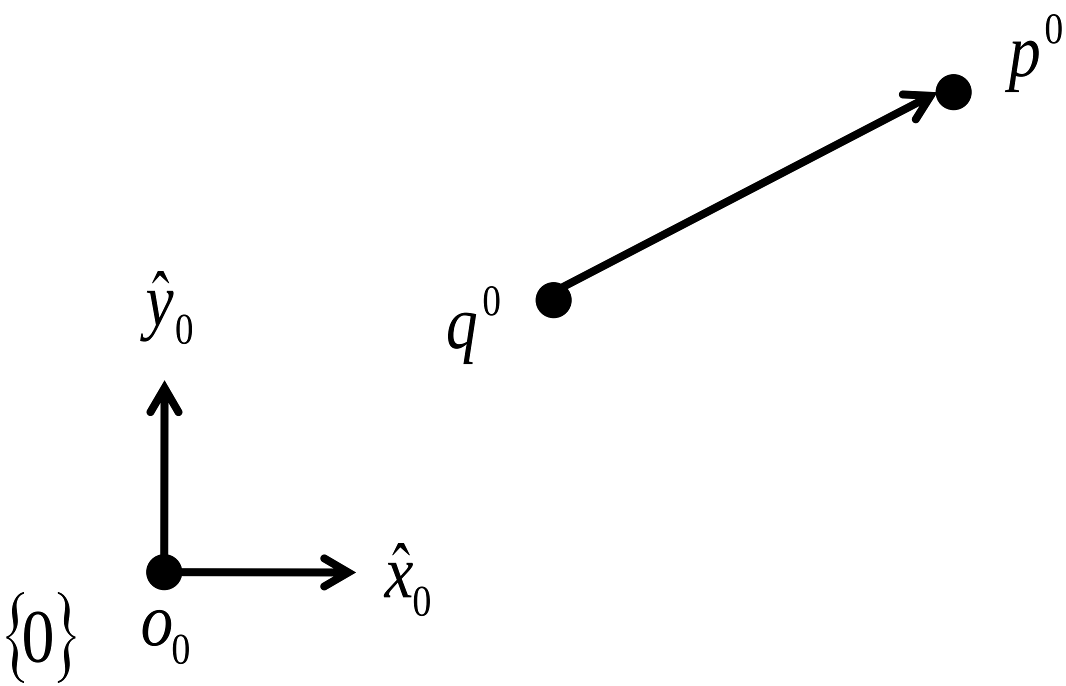
Points and Vectors

- point : a location in space
- vector : magnitude (length) and direction between two points



Coordinate Frames

- choosing a frame (a point and two perpendicular vectors of unit length) allows us to assign coordinates



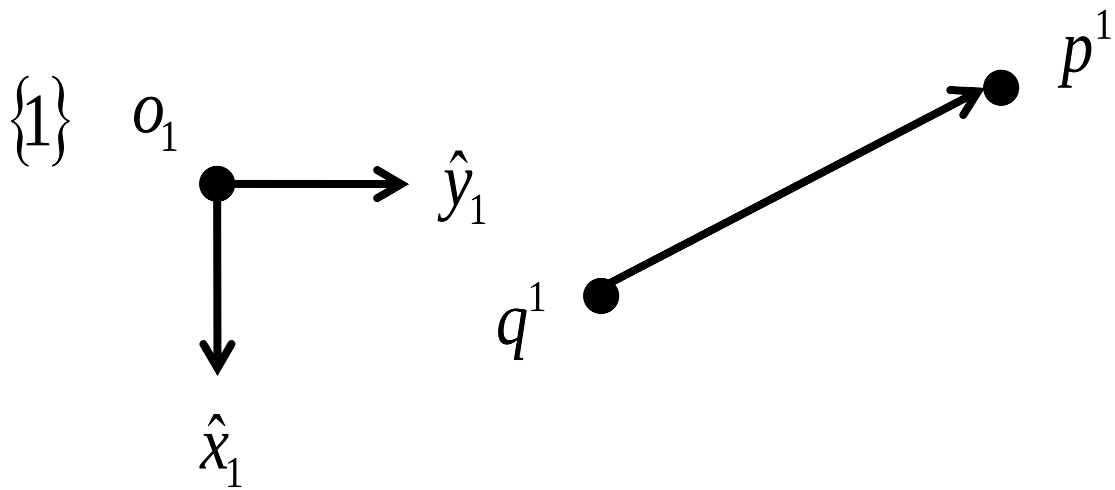
$$p^0 = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

$$q^0 = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$$

$$v^0 = p^0 - q^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Coordinate Frames

- the coordinates change depending on the choice of frame



$$p^1 = \begin{bmatrix} -0.5 \\ 4 \end{bmatrix}$$

$$q^1 = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}$$

$$v^1 = p^1 - q^1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

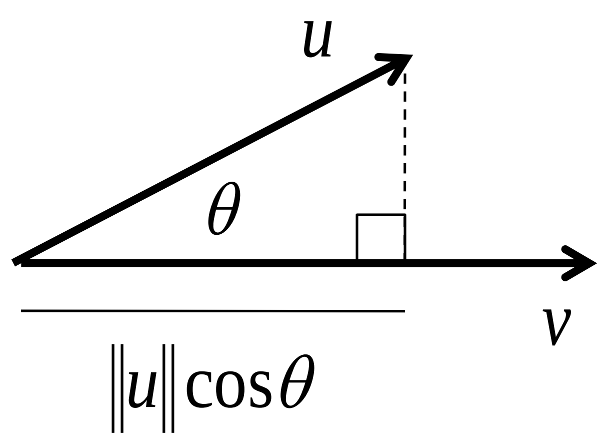
Dot Product

- the dot product of two vectors

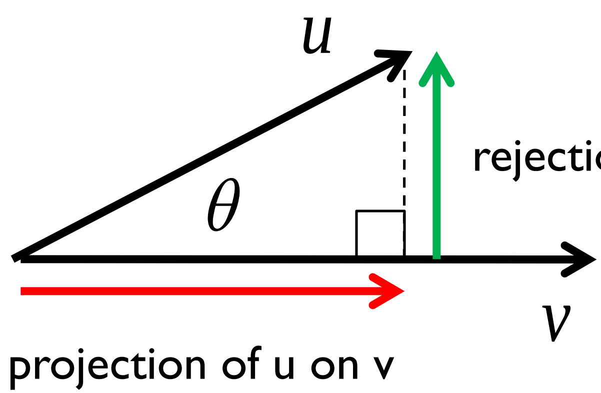
$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = u^T v$$

$$u \cdot v = \|u\| \|v\| \cos\theta$$



Vector Projection and Rejection



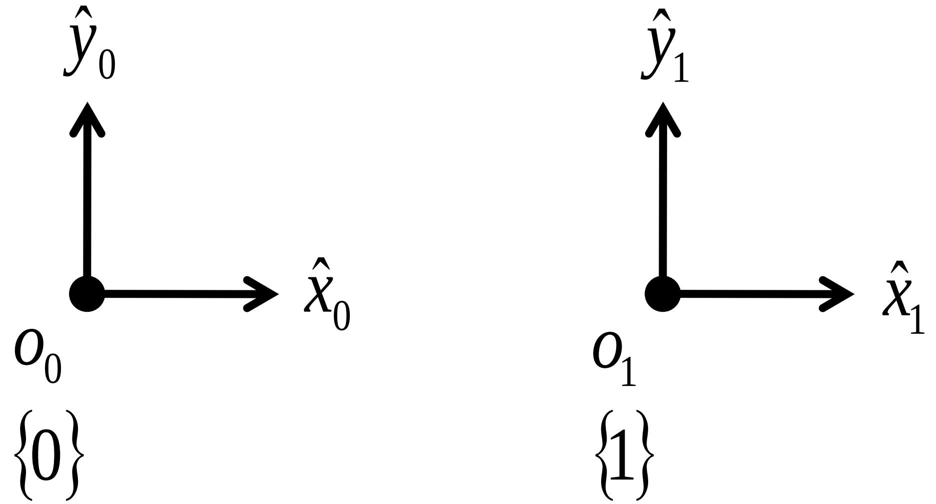
$$u - \frac{u \cdot v}{v \cdot v} v$$

$$\frac{u \cdot v}{v \cdot v} v$$

- if u and v are unit vectors (have magnitude equal to 1) then the projection becomes

$$\hat{u} \cdot \hat{v}\, \hat{v}$$

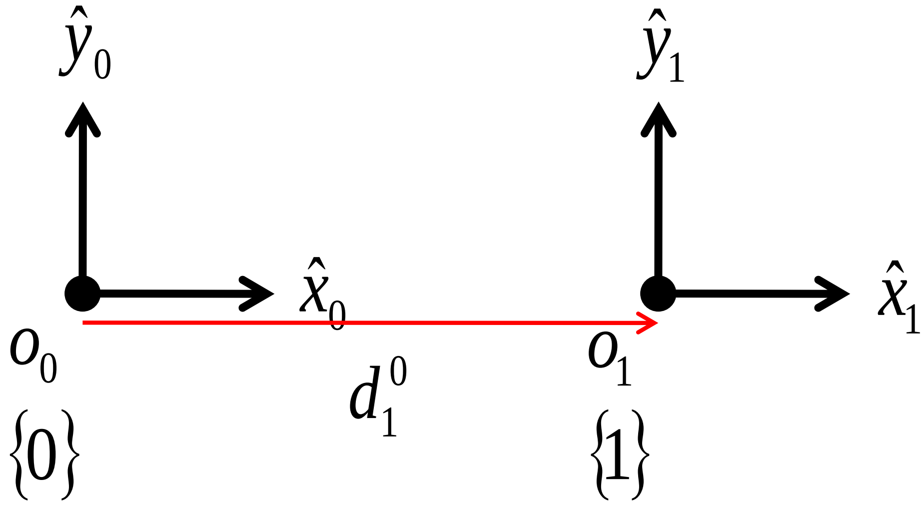
Translation



- suppose we are given o_1 expressed in $\{0\}$

$$o_1^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Translation 1



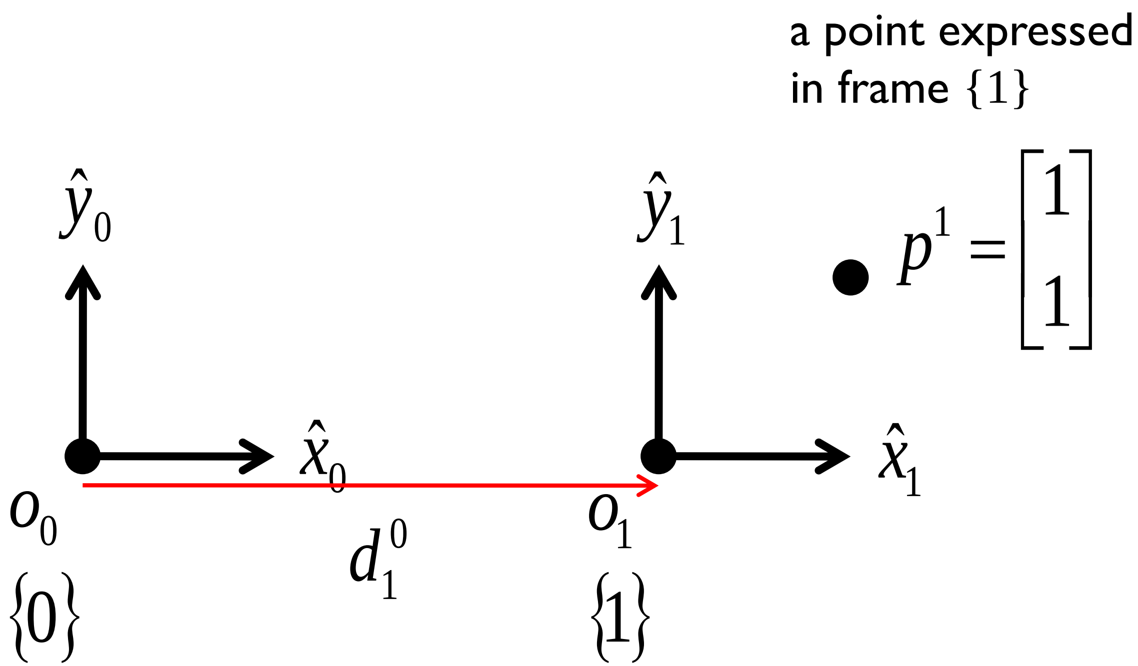
- the location of $\{1\}$ expressed in $\{0\}$

$$d_1^0 = o_1^0 - o_0^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Translation 1

1. the translation vector d_j^i can be interpreted as the location of frame $\{j\}$ expressed in frame $\{i\}$

Translation 2



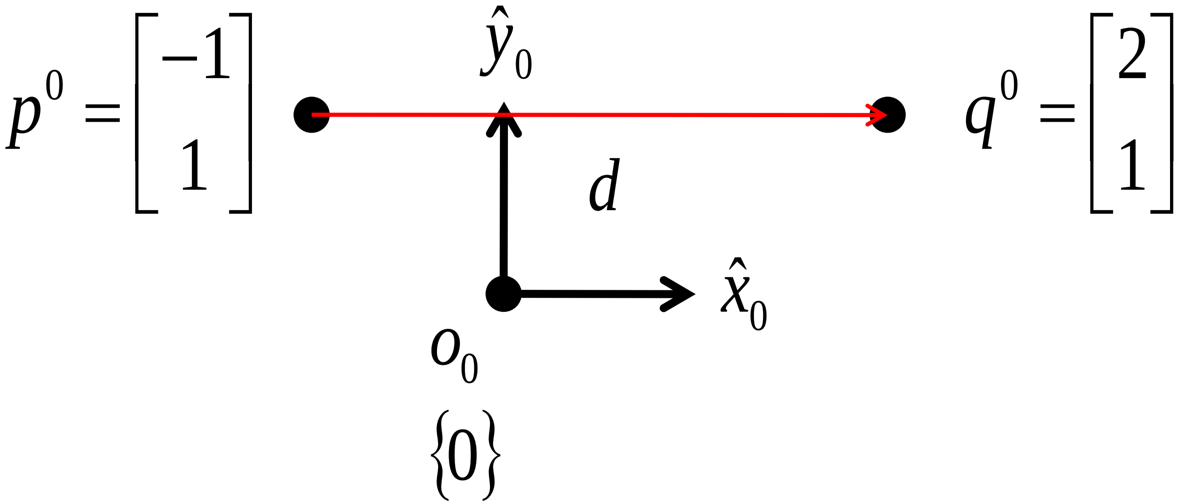
- p^1 expressed in $\{0\}$

$$p^0 = d_1^0 + p^1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Translation 2

2. the translation vector d_j^i can be interpreted as a coordinate transformation of a point from frame {j} to frame {i}

Translation 3



- q^0 expressed in $\{0\}$

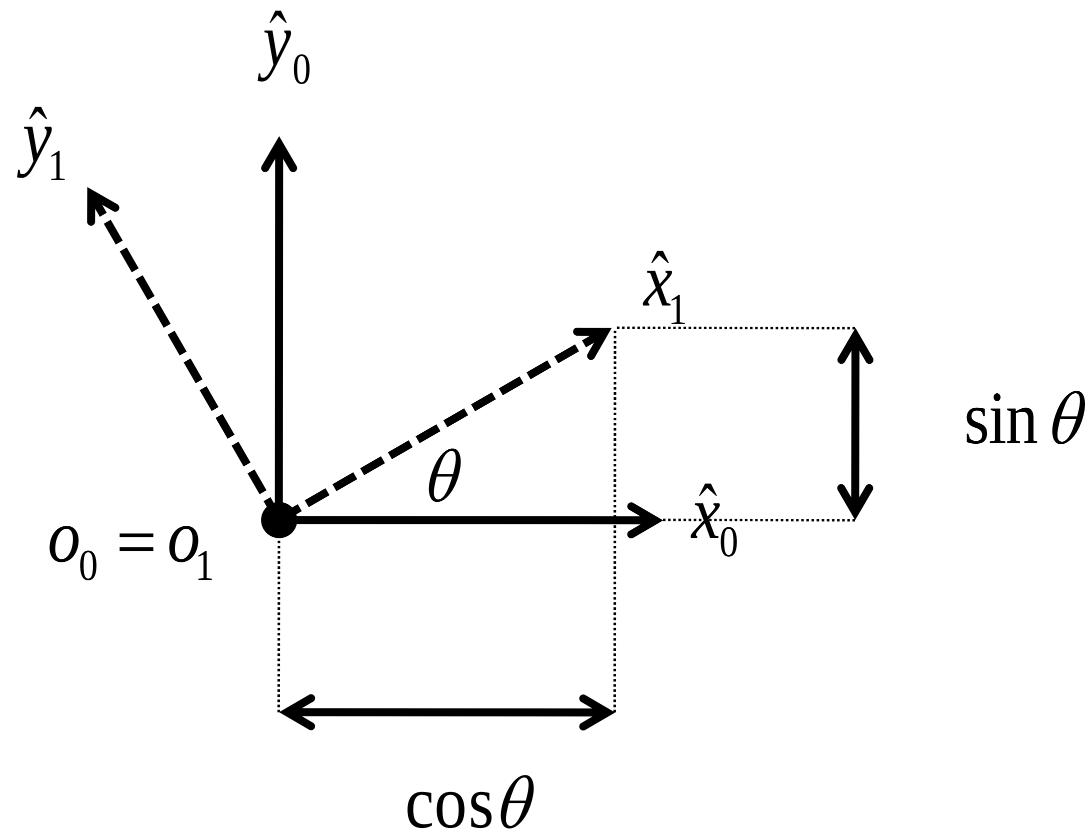
$$q^0 = d + p^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Translation 3

3. the translation vector d can be interpreted as an operator that takes a point and moves it to a new point in the same frame

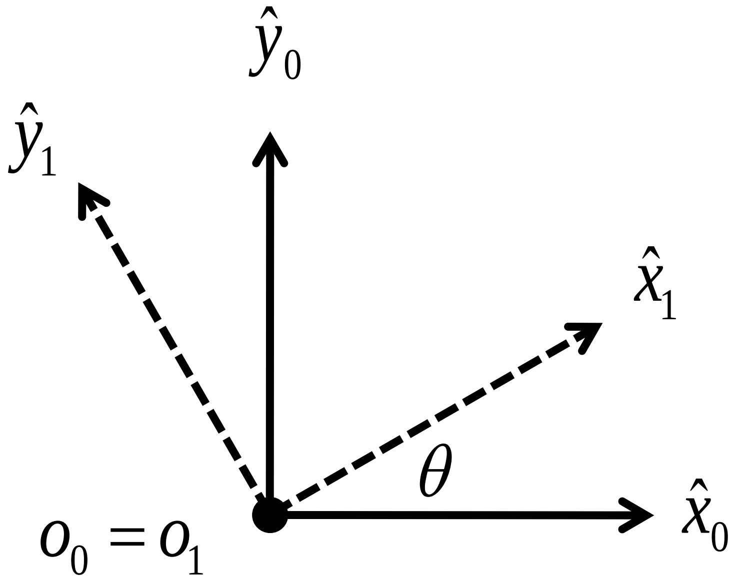
Rotation

- suppose that frame *{1}* is rotated relative to frame *{0}*



Rotation 1

- the orientation of frame {1} expressed in {0}



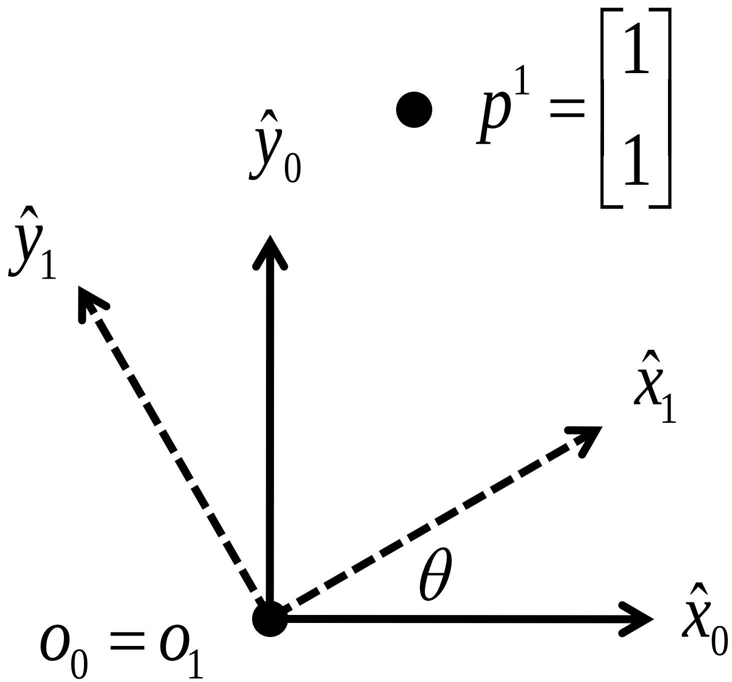
$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$

Rotation 1

1. the rotation matrix R_j^i can be interpreted as the orientation of frame {j} expressed in frame {i}

Rotation 2

- p^1 expressed in $\{0\}$



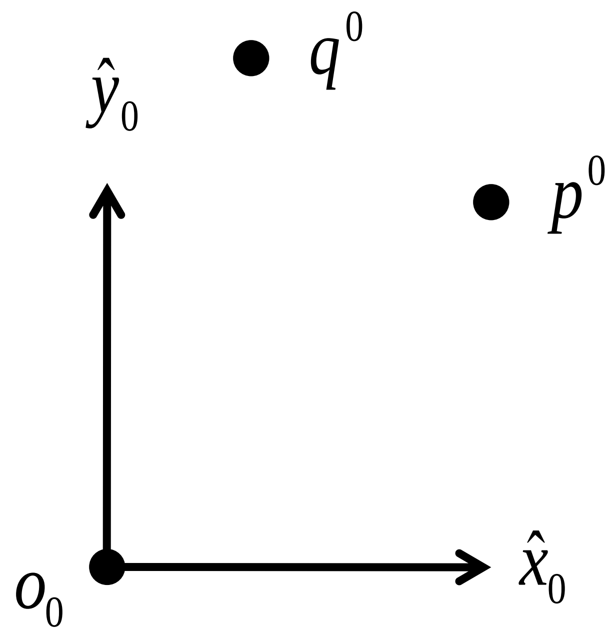
$$p^0 = R_1^0 p^1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Rotation 2

2. the rotation matrix R_j^i can be interpreted as a coordinate transformation of a point from frame {j} to frame {i}

Rotation 3

- q^0 expressed in $\{0\}$



$$q^0 = R\ p^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

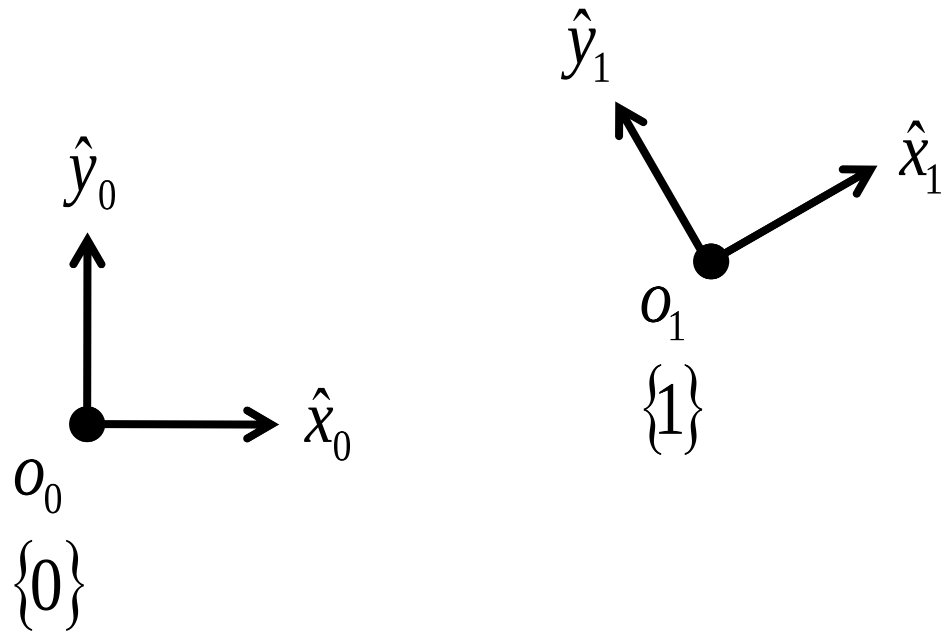
Rotation 3

3. the rotation matrix R can be interpreted as an operator that takes a point and moves it to a new point in the same frame

Properties of Rotation Matrices

- $R^T = R^{-1}$
- the columns of R are mutually orthogonal
- each column of R is a unit vector
- $\det R = 1$ (the determinant is equal to 1)

Rotation and Translation



Rotations in 3D

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$